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Uniform Rotundity of Musielak–Orlicz Sequence Spaces

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We present a criterion for uniform rotundity of Musielak–Orlicz sequence spaces. In particular, we get a better characterization of uniform rotundity of Banach spaces $l(\{p_i\})$, called Nakano spaces, considered by K. Sundaresan (*Studia Math.* **39** (1971), 227–331. © 1986 Academic Press, Inc.

INTRODUCTION

Geometrical properties of Banach spaces play an important role in the theory of approximation and optimization. The property of uniform rotundity ensures, for example, the existence and unicity of nearest points in best approximation problems. Moreover, uniformly rotund Banach spaces are E-spaces where "all convex norm-minimization problems are 'strongly solvable' and all convex best approximation problems are 'well posed' in the sense of Hadamard" [4]. Among the many papers concerning approximation problems, some, e.g., [3, 10], deal with best approximation in Orlicz spaces. It is important there to know how rotundity of Orlicz space is expressed in terms of Young functions. So it seems worthwhile to look for criteria for the validity of various geometrical properties in spaces of Orlicz type.

We know a criterion for uniform rotundity of Orlicz sequence space [8] and a sufficient condition and a little weaker necessary one for this property in Nakano space [12]. The Nakano spaces, like the Orlicz spaces, are particular cases of more general Musielak–Orlicz spaces. Here we will find necessary and sufficient conditions stated in terms of Young functions for uniform rotundity of such spaces. In particular, we get a criterion for the validity of this property in Nakano spaces.

Now we introduce the basic notations and definitions. In the following, let $\mathbb{R}$ be the real line, $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{N}$ the set of natural numbers. For arbitrary $a, b \in \mathbb{R}$, we write $\min(a, b) = a \wedge b$, $\max(a, b) = a \vee b$. Let

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$\varphi = (\varphi_n)$, where φ_n are Young functions, i.e., $\varphi_n: \mathbb{R}_+ \to \mathbb{R}_+$ are convex and $\varphi_n(0) = 0$ for all $n \in \mathbb{N}$. Let $\varphi_n^{-1}: \mathbb{R}_+ \to \mathbb{R}_+$ be the generalized inverse function, i.e., $\varphi_n^{-1}(v) = \inf\{u \geqslant 0: \varphi_n(u) > v\}$. The Musielak–Orlicz space l_φ is the set of all real sequences $x = (u_n)$ such that

$$I_\varphi(\lambda x) = \sum_{n=1}^{\infty} \varphi_n(\lambda |u_n|) = \sum_{\mathbb{N}} \varphi_n(\lambda |u_n|) = \sum_n \varphi_n(\lambda |u_n|) < \infty$$

for some $\lambda > 0$ dependent on x. If all functions φ_n are identical then l_φ becomes an usual Orlicz space. Here, l_φ is endowed with Luxemburg norm, i.e., $\|x\| = \inf\{\varepsilon > 0: I_\varphi(x/\varepsilon) \leqslant 1\}$ (for details of Musielak–Orlicz spaces see [11]). Let us define a new function $\psi = (\psi_n)$ as follows,

$$\begin{aligned} \psi_n(u) &= \varphi_n(b_n u) && \text{if } 0 \leqslant u \leqslant 1 \\ &= u && \text{if } u > 1, \end{aligned}$$

where $\varphi_n(b_n) = 1$. The spaces l_φ and l_ψ are isometrically equal. Indeed, let $T: l_\varphi \to l_\psi$ be such that $Tx = y$, where $y = (u_n/b_n)$ for $x = (u_n)$. If $\varepsilon > 0$ is such that $I_\psi(y/\varepsilon) \leqslant 1$ (which is equivalent to $I_\varphi(x/\varepsilon) \leqslant 1$) then $|u_n/b_n \varepsilon| \leqslant 1$ and so $I_\psi(y/\varepsilon) = \sum_{n=1}^\infty \psi_n(|u_n/b_n \varepsilon|) = \sum_{n=1}^\infty \varphi_n(|u_n/\varepsilon|) = I_\varphi(x/\varepsilon)$. It means that $\|y\|_\psi = \|x\|_\varphi$, where $\|\ \|_\psi$ ($\|\ \|_\varphi$) denotes the Luxemburg norm in $l_\psi(l_\varphi)$.

Henceforth, by virtue of the above considerations, we assume that $\varphi_n(1) = 1$, $M = \sup_n \varphi_n(2) < \infty$, and φ_n are convex on the interval $[0, 1]$ and are nondecreasing on $\mathbb{R}_+$. However, we must remember that φ_n may be not convex on the whole set $\mathbb{R}_+$. Now, define a few conditions concerning the function φ.

It is said that φ satisfies the condition δ_2 [7] if there exist constants $k, \delta > 0$ and a nonnegative sequence $(c_n) \in l_1$ such that

$$\varphi_n(2u) \leqslant k\varphi_n(u) + c_n \tag{0.1}$$

for each $n \in \mathbb{N}$ and $u \in \mathbb{R}_+$ when $\varphi_n(u) \leqslant \delta$. It is not difficult to show that under additional assumptions made on φ, the condition δ_2 is fulfilled iff there exists a nonnegative sequence $(c_n) \in l_1$ such that the inequality (0.1) is fulfilled for each $n \in \mathbb{N}$ and all $u \in (0, 1)$. Indeed, if $\delta < \varphi_n(u) \leqslant 1$ then $\varphi_n(2u) \leqslant M = (M/\delta)\, \delta \leqslant (M/\delta)\, \varphi_n(u)$. Thus $\varphi_n(2u) \leqslant (k \vee M/\delta)\, \varphi_n(u) + c_n$ for all $u \leqslant 1$. We also note that φ satisfies the condition δ_2 iff there are a constant k and a nonnegative sequence (c_n) such that

$$\sum_{n=1}^{\infty} \varphi_n(c_n) < \infty \qquad \text{and} \qquad \varphi_n(2u) \leqslant k\varphi_n(u) \tag{0.2}$$

for $u \in [c_n, 1]$, $n \in \mathbb{N}$. If, in addition, each φ_n vanishes only at zero, then, for each $\varepsilon > 0$, a sequence (c_n) and a constant k in (0.2) may be chosen in such a way that

$$\sum_{n=1}^{\infty} \varphi_n(c_n) < \varepsilon. \tag{0.3}$$

We say that φ satisfies the condition (*) if for each $\varepsilon \in (0.1)$ there exists $\delta > 0$ such that $\varphi_n(u) < 1 - \varepsilon$ implies $\varphi_n((1+\delta)u) \leqslant 1$ for all $u \in \mathbb{R}_+$, $n \in \mathbb{N}$.

Let us introduce a function h: $\mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty)$ in the following way,

$$\begin{aligned} h(u, v) &= 2\Phi((u+v)/2)/(\Phi(u)+\Phi(v)) && \text{if} \quad \Phi(u) \vee \Phi(v) > 0 \\ &= 0 && \text{if} \quad \Phi(u) \vee \Phi(v) = 0 \end{aligned}$$

for an arbitrary Young function Φ. If, in particular, Φ is equal to φ_n then we will denote the function h by h_n.

Let d be positive number. It is said that φ is uniformly convex in the d-neighbourhood of zero if for each $a \in [0, 1)$ there exist $\delta \in (0, 1)$ and a non-negative sequence (d_n) such that $\varphi_n(d_n) \leqslant d$ and

$$\sum_{n=1}^{\infty} \varphi_n(d_n) < \infty \qquad \text{and} \qquad h_n(u, au) \leqslant 1 - \delta$$

for $u \in (d_n, \varphi_n^{-1}(d)]$, $n \in \mathbb{N}$. Recall that a Young function Φ is strictly convex on an interval $[a, b]$ if $\Phi((u+v)/2) < (\Phi(u)+\Phi(v))/2$ for every $u, v \in [a, b]$, $u \neq v$. A Banach space $(X, \| \|)$ is said to be uniformly rotund [2] if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\|x\| = 1$, $\|y\| = 1$, and $\|x - y\| \geqslant \varepsilon$ then $\|(x+y)/2\| \leqslant 1 - \delta(\varepsilon)$ (equivalently we can put $\|x\| \leqslant 1$, $\|y\| \leqslant 1$ instead of $\|x\| = 1$, $\|y\| = 1$). Similarly, it is said that the modular I_φ is uniformly rotund if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $I_\varphi(x) = 1$, $I_\varphi(y) = 1$, and $I_\varphi(x - y) \geqslant \varepsilon$ then $I_\varphi((x+y)/2) \leqslant 1 - \delta(\varepsilon)$.

We give the following known results, needed in the sequel, for completeness.

0.1. THEOREM. (a) [6] *The norm and modular convergence are equivalent in* l_φ, i.e., $\|x\|_\varphi \to 0 \Leftrightarrow I_\varphi(\lambda x) \to 0$ *for some* $\lambda > 0$, *iff the function* φ *satisfies the condition* δ_2 *and each* φ_n *vanishes only at zero.*

(b) [7] *We have an equivalence* $\|x\| = 1 \Leftrightarrow I_\varphi(x) = 1$ *iff the function* φ *satisfies the condition* δ_2.

0.2. THEOREM [7]. *The space* l_φ *is rotund iff the following conditions are satisfied:*

—*the function* φ *fulfills the condition* δ_2,

—*there exists a sequence* (a_n) *such that* $a_n \in [0, 1]$,

$\varphi_n(a_n)+\varphi_m(a_m)\geqslant 1$ for $n\neq m$ and each φ_n is strictly convex on $[0, a_n]$,

—each function φ_n vanishes only at zero.

0.3. LEMMA [9]. *The function h has the following properties:*

(1) $h(u, v)=h(v, u)$,

(2) *a function $u\to h(u, v)$ is nondecreasing on an interval $[0, v]$ for each $v\in\mathbb{R}_+$.*

0.4. LEMMA [9]. *If Φ is a strictly convex Young function on an interval $[0, a]$ then for every $\varepsilon>0$, $d_1, d_2\in(0, a]$, $d_1<d_2$, there exists $p=p(\varepsilon, d_1, d_2)\in(0, 1)$ such that*

$$h(u, v)\leqslant 1-p$$

if $|u-v|\geqslant\varepsilon(u\vee v)$ and $u\vee v\in[d_1, d_2]$.

RESULTS

1. LEMMA. *If φ satisfies the condition (*) then there exists $r_0\in(0, 1)$ such that* $\inf_n\varphi_n(r_0)=M_0>0$.

Proof. Suppose, to the contrary, $\inf_n\varphi_n(r)=0$ for every $r\in(0, 1)$. Then there exists $m_n\in\mathbb{N}$ such that $\varphi_{m_n}(1-1/n)<1/2$ for every $n\in\mathbb{N}$. Hence, by the condition (*), we have

$$\varphi_{m_n}((1+\delta)(1-1/n))\leqslant 1 \tag{1.1}$$

for all $n\in\mathbb{N}$ and some $\delta\in(0, 1)$. But $(1+\delta)(1-1/n)>1$ for sufficiently large n, i.e., $\varphi_i((1+\delta)(1-1/n))>1$ for every i, which contradicts (1.1).

2. LEMMA. *If φ satisfies the conditions (*) and δ_2 and each φ_n vanishes only at zero then* $\inf_n\delta_n(r)>0$ *for every $r\in(0, 1)$.*

Proof. We have $M_0=\inf_n\varphi_n(r_0)>0$ for some $r_0\in(0, 1)$, by the previous lemma. But

$$\varphi_n(r_0/2)\geqslant(1/k)(\varphi_n(r_0)-c_n)\geqslant(1/k)(M_0-c_n),$$

by the condition δ_2. We can choose n_1 such that $\inf_{n>n_1}(M_0-c_n)>0$, because $c_n\to 0$. Putting

$$M_1=\inf_{n>n_1}(1/k)(M_0-c_n)\wedge\inf_{1\leqslant n\leqslant n_1}\varphi_n(r_0/2)$$

we have $\inf_n \varphi_n(r_0/2) \geqslant M_1 > 0$. Similarly it can be shown that $\inf_n \varphi_n(r_0/2^i) > 0$ for each natural number i. By virtue of monotonicity of φ_n this ends the proof.

3. LEMMA. *If φ satisfies the condition δ_2 then the family $\{\varphi_n\}$ is equicontinuous on the interval* $[0, 1]$: i.e., *for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|u - v| < \delta$ implies $|\varphi_n(u) - \varphi_n(v)| < \varepsilon$ for all $u, v \in [0, 1]$, $n \in \mathbb{N}$.*

Proof. For a contrary there exists $\varepsilon > 0$, sequences (u_m), $(v_m) \subset [0, 1]$ and a subsequence (n_m) of natural numbers such that

$$|u_m - v_m| < 1/m \qquad \text{and} \qquad |\varphi_{n_m}(u_m) - \varphi_{n_m}(v_m)| > \varepsilon \tag{3.1}$$

for every $m \in \mathbb{N}$. Since the functions φ_n are uniformly continuous on $[0, 1]$ we can assume without loss of generality that $n_1 < n_2 < \cdots$. Sequences (u_m), (v_m) must possess accumulation points. Let us note that the accumulation points of (u_m) and $v_m)$ are equal: this results simply from (3.1).

First, let the number 1 be a point of accumulation of (u_m) and (v_m). Assume for simplicity that $u_m \to 1$ and $v_m \to 1$ and

$$\varphi_{n_m}(u_m) \geqslant \varphi_{n_m}(v_m) \tag{3.2}$$

for all $m \in \mathbb{N}$. Then $\varphi_{n_m}(v_m) \leqslant \varphi_{n_m}(u_m) - \varepsilon \leqslant 1 - \varepsilon$, by (3.1). Hence and by the assumed condition (*) we have $\varphi_{n_m}((1+\delta) v_m) \leqslant 1$ for each $m \in \mathbb{N}$ and some $\delta > 0$. So $(1+\delta) v_m \leqslant 1$ for every $m \in \mathbb{N}$, which contradicts $v_m \to 1$.

Now, let 0 be a point of accumulation of (u_m) and (v_m). Suppose $u_m \to 0$, $v_m \to 0$, and the inequality (3.2) holds. Also, let us note that $\varphi_n(u) \leqslant u$ for all $u \in [0, 1]$. Therefore and by virtue of (3.1) we have $\varepsilon < \varepsilon + \varphi_{n_m}(v_m) < \varphi_{n_m}(u_m) \leqslant u_m$ for every $m \in \mathbb{N}$, which contradicts $u_m \to 0$.

Finally, let $s \in (0, 1)$ be a point of accumulation of (u_m) and (v_m). Taking $\alpha \in (0, s \wedge (1-s))$ we have

$$(\varphi_n(s+\alpha) - \varphi_n(s))/\alpha \leqslant (1 - \varphi_n(s))/(1-s),$$
$$(\varphi_n(s) - \varphi_n(s-\alpha))/\alpha \leqslant (1 - \varphi_n(s-\alpha))/(1-(s-\alpha)),$$

by convexity of φ_n. Hence

$$\begin{aligned} \varphi_n(s+\alpha) &\leqslant \varphi_n(s) + \alpha/(1-s), \\ \varphi_n(s) &< \varphi_n(s-\alpha) + \alpha/(1-s), \end{aligned} \tag{3.3}$$

for each $n \in \mathbb{N}$. However, $u_m, v_m \in [s-\alpha, s+\alpha]$ for infinitely many indices. Therefore

$$|\varphi_{n_m}(u_m) - \varphi_{n_m}(v_m)| \leqslant \varphi_{n_m}(s+\alpha) - \varphi_{n_m}(s-\alpha) \leqslant 2\alpha/(1-s)$$

for infinitely many indices m and $\alpha \in (0, s \wedge (1-s))$, by (3.3) and the monotonicity of φ_n. Taking $\alpha < \varepsilon(1-s)/2$ we get a contradiction with (3.1).

4. LEMMA. *The function φ is uniformly convex in the d-neighbourhood of zero iff for each $\varepsilon \in (0, 1)$ there exist $p \in (0, 1)$ and a nonnegative sequence (d_n) such that $\varphi_n(d_n) \leqslant d$, $\sum_{n=1}^{\infty} \varphi_n(d_n) < \infty$, and*

$$h_n(u, v) \leqslant 1 - p$$

if $|u-v| \geqslant \varepsilon(u \vee v)$ and $u \vee v \in (d_n, \varphi_n^{-1}(d)]$, $n \in \mathbb{N}$.

Proof. Let (d_n) and δ be a sequence and a constant from the definition of uniform convexity of φ, chosen for $a = 1-\varepsilon$. Let u, v satisfy the assumptions of the lemma and let $u > v$. Then $u \in (d_n, \varphi_n^{-1}(d)]$ and $(1-\varepsilon) u \geqslant v$. Hence $h_n(u, v) \leqslant h_n(u, (1-\varepsilon) u)$, by the property (2) of h_n in Lemma 0.3. Thus, we have $h_n(u, v) \leqslant 1-p$, by uniform convexity of φ, putting $p = \delta$. The converse is immediate if we apply the inequality $h_n(u, v) \leqslant 1-p$ for $\varepsilon = 1-a$ and $v = au$, where $u \in (d_n, \varphi_n^{-1}(d)]$, $a \in [0, 1)$.

5. LEMMA. *If φ is uniformly convex in the d-neighbourhood of zero and each φ_n is strictly convex on the interval $[0, \varphi_n^{-1}(d)]$, respectively, then for every $\varepsilon \in (0, 1)$ there exists $\bar{p} \in (0, 1)$ and a nonnegative sequence $(\bar{d}_n)$ with $\sum_{n=1}^{\infty} \varphi_n(\bar{d}_n) < \varepsilon$ and such that the previous lemma holds with $\bar{p}$ and $(\bar{d}_n)$ instead of p and (d_n).*

Proof. We can assume that $\varepsilon < d$. Let (d_n) and p be as in the previous lemma. We have $\sum_{n=n_0+1}^{\infty} \varphi_n(d_n) < \varepsilon/2$ for some $n_0 \in \mathbb{N}$. Let a_n be positive numbers such that $\sum_{n=1}^{n_0} \varphi_n(a_n) < \varepsilon/2$. Since φ_n are strictly convex on $[0, \varphi_n^{-1}(d)]$, so $h_n(u, v) \leqslant 1-p_n$ for some $p_n \in (0, 1)$ if $|u-v| \geqslant \varepsilon(u \vee v)$ and $u \vee v \in (a_n, \varphi_n^{-1}(d)]$ for $n = 1, \ldots, n_0$, by Lemma 0.4. Putting

$$\begin{aligned} \bar{d}_n &= d_n \qquad \text{if} \quad n = n_0+1, n_0+2, \ldots \\ &= a_n \qquad \text{if} \quad n = 1, \ldots, n_0, \end{aligned}$$

and $\bar{p} = p_1 \vee p_2 \vee \cdots \vee p_{n_0} \vee p$, Lemma 4 holds with $\bar{p}$ and $(\bar{d}_n)$ in place of p and (d_n).

6. LEMMA. *If φ is uniformly convex in the d-neighbourhood of zero, φ satisfies the condition δ_2 and each φ_n is strictly convex on the interval $[0, \varphi_n^{-1}(d)]$, respectively, then for every $\varepsilon \in (0, 1)$ there exist $k > 0$, $p \in (0, 1)$ and a nonnegative sequence (c_n) such that $\sum_{n=1}^{\infty} \varphi_n(2c_n) < \varepsilon$ and*

$$\varphi_n(2u) \leqslant k\varphi_n(u)$$

for $u \in [c_n, 1]$ *and*

$$h_n(u, v) \leqslant 1 - p$$

if $|u - v| \geqslant \varepsilon(u \vee v)$ *and* $u \vee v \in (c_n, \varphi_n^{-1}(d)]$, $n \in \mathbb{N}$.

Proof. Functions φ_n are strictly convex on some neighbourhood of zero, so they vanish only at zero. Hence and by the supposed condition δ_2 (see also (0.3)) it follows the existence of a sequence (c_n') and a constant $k > 0$ such that $\varphi_n(2u) \leqslant k\varphi_n(u)$ for $u \in [c_n', 1]$, where $\sum_{n=1}^{\infty} \varphi_n(2c_n') < \varepsilon/2$. Moreover, we have $\sum_{n=1}^{\infty} \varphi_n(2\bar{d}_n) < \infty$ for a sequence $(\bar{d}_n)$ from the previous lemma, by the condition δ_2. Acting in a manner similar to that in the preceding proof, we modify $(\bar{d}_n)$ in such a way that $\sum_{n=1}^{\infty} \varphi_n(2\bar{d}_n) < \varepsilon/2$. Putting $c_n = c_n' \vee \bar{d}_n$ we end the proof of the lemma.

7. Lemma. *If* φ *is uniformly convex in the d-neighbourhood of zero and satisfies conditions* δ_2 *and* (*), *and each* φ_n *is strictly convex on* $[0, \varphi_n^{-1}(d)]$, *then for arbitrary* $\alpha, \beta \in [0, 1]$ *satisfying the inequality* $0 \leqslant \alpha < \gamma = \beta \wedge d$ *there exists* $p \in (0, 1)$ *such that*

$$h_n(u, v) \leqslant 1 - p$$

for every $n \in \mathbb{N}$, $u, v \in \mathbb{R}_+$ *if* $0 \leqslant u \leqslant \varphi_n^{-1}(\alpha)$ *and* $\varphi_n^{-1}(\gamma) \leqslant v \leqslant 1$.

Proof. Let $\varphi_n(u_n) = \alpha$, $\varphi_n(v_n) = \gamma$. Since $\varphi_n(v_n) - \varphi_n(u_n) = \gamma - \alpha > 0$, there exists $\delta_0 \in (0, \alpha)$ such that $v_n - u_n > \delta_0$ for every $n \in \mathbb{N}$, by Lemma 3. Hence $v_n - u_n \geqslant \delta_0 v_n$, because $v_n \in (0, 1]$. Applying Lemma 5 with δ_0 in place of ε we find a nonnegative sequence (d_n) and a constant $q \in (0, 1)$ such that $\sum_{n=1}^{\infty} \varphi_n(d_n) \leqslant \delta_0 < \alpha < \gamma$ and

$$h_n(u_n, v_n) \leqslant 1 - q \tag{7.1}$$

for each $n \in \mathbb{N}$, because $v_n - u_n \geqslant \delta_0 v_n$ and $\varphi_n(u_n) \vee \varphi_n(v_n) = \gamma \in (\varphi_n(d_n), d]$. Let $v \in [\varphi_n^{-1}(\gamma), 1]$. We have the inequalities

$$\varphi_n((u_n + v)/2) \leqslant \frac{\varphi_n(v) - \varphi_n((u_n + v_n)/2)}{v - (u_n + v_n)/2} ((u_n + v)/2 - v) + \varphi_n(v),$$

$$(\varphi_n(u_n) + \varphi_n(v))/2 \geqslant \frac{\varphi_n(v) - (\varphi_n(u_n) + \varphi_n(v_n))/2}{v - (u_n + v_n)/2} ((u_n + v)/2 - v) + \varphi_n(v),$$

by the convexity of φ_n. Hence and by (7.1) we get

$$h_n(u_n, v) \leqslant (a_n + (1 - q)(\alpha + \gamma)/2)/(a_n + (\alpha + \gamma)/2)$$

for each $n \in \mathbb{N}$, where $a_n = \varphi_n(v)(v - v_n)/(v - u_n) \in (0,1)$. Since the function

$u \to (u+b_1)/(u+b_2)$, $b_1 < b_2$, is increasing for $u \in [0, 1]$, so $h_n(u_n, v) \leqslant 1-p$, where $p = (q(\alpha+\gamma)/2)/(1+(\alpha+\gamma)/2)$. Hence and by the second property of h_n we obtain

$$h_n(u, v) \leqslant 1-p$$

for all $n \in \mathbb{N}$, if $\varphi_n(u) \leqslant \alpha$ and $\gamma \leqslant \varphi_n(v) \leqslant 1$, which ends the proof.

1.1. *Remark.* (1) It is not difficult to show that uniform convexity of φ in the d-neighbourhood of zero implies this in the c-neighbourhood of zero for $c \in (0, d]$.

(2) Let N be a subset of $\mathbb{N}$. We say that a family $(\varphi_n)_{n \in N}$ is uniformly convex in the d-neighbourhood of zero, if the function $\psi = (\psi_n)$ has this property, where $\psi_n = \varphi_n$ for $n \in N$ and $\psi_n = 0$ for $n \notin N$. In Lemmas 4–7 we can replace the function φ by a family $(\varphi_n)_{n \in N}$, obtaining the statements of the lemmas not for all $n \in \mathbb{N}$ but only for $n \in N$.

8. LEMMA. *Let $(X, \| \ \|)$ be a normed space. If $f: X \to \mathbb{R}$ is a convex function in the set $K(0, 1) = \{x \in X: \|x\| \leqslant 1\}$ and $|f(x)| \leqslant M$ for all $x \in K(0, 1)$ and some $M > 0$ then f is almost uniformly continuous in $K(0, 1)$;* i.e., *for all $d \in (0, 1)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\|y\| \leqslant d$ and $\|x-y\| < \delta$ implies $|f(x)-f(y)| < \varepsilon$ for all $x, y \in K(0, 1)$.*

Proof. We can always suppose that $M \geqslant 1$. Let $g_y(x) = f(x+y) - f(y)$. It is enough to show uniform continuity of this function at zero with respect to $y \in K(0, d)$. Note that $g_y(0) = 0$ and the function $g_y(x)$ is convex for such arguments x for which $\|x+y\| \leqslant 1$. If $\|y\| \leqslant d$ and $\|x\| \leqslant 1-d$ then $\|x+y\| \leqslant 1$ and so $|g_y(x)| \leqslant |f(x+y)| + |f(y)| \leqslant 2M$. Putting $\delta = (1-d)\,\varepsilon/2M$ and taking $y \in K(0, d)$ and $x \in K(0, \delta)$, we have

$$g_y(x) \leqslant (\varepsilon/2M)\, g_y(2Mx/\varepsilon) \leqslant (\varepsilon/2M)\, 2M = \varepsilon, \tag{8.1}$$

because $\|2Mx/\varepsilon\| \leqslant 1-d$ for $x \in K(0, \delta)$. Moreover $0 = g_y(0) \leqslant (1/(1+\varepsilon/2M))\, g_y(x) + ((\varepsilon/2M)/(1+\varepsilon/2M))\, g_y(-2Mx/\varepsilon)$, which implies

$$g_y(x) \geqslant (-\varepsilon/2M)\, g_y(-2Mx/\varepsilon) \geqslant -\varepsilon, \tag{8.2}$$

because $\|-2Mx/\varepsilon\| \leqslant 1-d$ for $x \in K(0, \delta)$. The inequalities (8.1) and (8.2) end the proof.

9. LEMMA. *If the condition δ_2 is fulfilled then the following conditions are equivalent*:

(1) *the function φ satisfies the condition* (*),

(2) *for every $\varepsilon \in (0, 1)$ there exists $\eta \in (0, 1)$ such that the inequality $I_\varphi(x) \leqslant 1-\varepsilon$ implies $\|x\| \leqslant 1-\eta$ for $x \in l_\varphi$.*

Proof. (1)$\Rightarrow$(2) Let $\varepsilon\in(0,1)$ be chosen arbitrarily and let $x=(u_n)$ be such that $I_\varphi(x)\leqslant 1-\varepsilon$. Then $\varphi_n(|u_n|)\leqslant 1-\varepsilon$ for each $n\in\mathbb{N}$. Hence $\varphi_n((1+\delta)|u_n|)\leqslant k\varphi_n(|u_n|)+c_n$, where k and (c_n) are the constant and the sequence from the condition δ_2. Therefore $I_\varphi((1+\delta)x)\leqslant P$, with $P=k+\sum_{n=1}^{\infty}c_n<\infty$. Let us introduce a set A and a function $g\colon \mathbb{R}_+\to[0,+\infty]$ in the following way,

$$A=\{x\in l_\varphi : I_\varphi(x)\leqslant 1-\varepsilon\},$$
$$g(\lambda)=\sup_{x\in A} I_\varphi(\lambda x),$$

for $\lambda\in\mathbb{R}_+$. The function g is convex, $g(0)=0$, $g(1)\leqslant 1-\varepsilon$, and $g(1+\delta)\leqslant P<\infty$. Hence it is continuous on the interval $[0,1+\delta]$. Thus there exists $\lambda_0\in(1,1+\delta]$ such that $g(\lambda_0)\leqslant 1$, by the Darboux property. It means that $I_\varphi(\lambda_0 x)\leqslant 1$ for all $x\in A$. Then, putting $\eta=1-1/\lambda_0$ we have $\|x\|\leqslant 1-\eta$ for each $x\in A$.

(2)$\Rightarrow$(1) For an arbitrary $\varepsilon\in(0,1)$ and $n\in\mathbb{N}$, let us take $u\in\mathbb{R}_+$ such that $\varphi_n(u)\leqslant 1-\varepsilon$. If we put $x=ue_n$ then $I_\varphi(x)=\varphi_n(u)\leqslant 1-\varepsilon$. So, there is $\eta\in(0,1)$ such that $\|x\|\leqslant 1-\eta$. Hence simply $I_\varphi(x/(1-\eta))=\varphi_n(u/(1-\eta))\leqslant 1$. Putting $\delta=\eta/(1-\eta)$ we get the condition (1).

10. Proposition. *The condition*

$$\text{for every } \varepsilon>0 \text{ there exists } \delta>0 \text{ such that } I_\varphi(x)\leqslant 1,\ I_\varphi(y)\leqslant 1 \text{ and } I_\varphi(x-y)\leqslant\delta \text{ imply } |I_\varphi(x)-I_\varphi(y)|<\varepsilon \text{ for } x,y\in l_\varphi \tag{10.1}$$

holds if and only if the function φ fulfills the conditions (*) and δ_2 and each φ_n vanishes only at zero.

Proof. Assume the condition (*) does not hold. Then there exist $\varepsilon>0$ and sequences $(\delta_m)=(1/m)$, (n_m), (u_m) such that $\varphi_{n_m}(u_m)\leqslant 1-\varepsilon$ and $\varphi_{n_m}((1+\delta_m)u_m)>1$. Without loss of generality, we can take $n_1<n_2<\cdots$. Let

$$x_m=u_m e_{n_m},\qquad y_m=(1+\alpha_m)u_m e_{n_m},$$

where $\alpha_m\in(0,\delta_m)$ is such that $\varphi_{n_m}((1+\alpha_m)u_m)=1$. We have $I_\varphi(x_m)<1$, $I_\varphi(y_m)=1$ and $I_\varphi(x_m-y_m)=\varphi_{n_m}(\alpha_m u_m)\leqslant\alpha_m(1-\varepsilon)\leqslant(1/m)(1-\varepsilon)\to 0$, when $m\to\infty$, because $0<\alpha_m<\delta_m=1/m$. However, $|I_\varphi(x_m)-I_\varphi(y_m)|=|\varphi_{n_m}(u_m)-\varphi_{n_m}((1+\alpha_m)u_m)|=1-\varphi_{n_m}(u_m)\geqslant\varepsilon$ for each $m\in\mathbb{N}$, which means that (10.1) is not fulfilled.

Now, suppose there exist $i\in\mathbb{N}$, $u_0\in(0,1)$ such that $\varphi_i(u_0)=0$. Let us take a number $u_1\in(1-u_0,1)$ and a sequence (u_m) such that $\varphi_{i+1}(u_m)\to 0$ when $m\to\infty$. Let

$$x_m=e_i,\qquad y_m=u_1e_i+u_me_{i+1}.$$

Then $I_\varphi(x_m)=1$, $I_\varphi(y_m)\leqslant 1$ for sufficiently large m and $I_\varphi(x_m-y_m)=\varphi_i(1-u_1)+\varphi_{i+1}(u_m)=\varphi_{i+1}(u_m)\to 0$, $m\to\infty$. However, $|I_\varphi(x_m)-I_\varphi(y_m)|=1-\varphi_i(u_1)-\varphi_{i+1}(u_m)\geqslant(1-\varphi_i(u_1))/2>0$ for large m, which means that (10.1) does not hold.

If the condition δ_2 is not fulfilled then there exists a sequence $(x_m)\subset l_\varphi$ such that $I_\varphi(x_m)\to 0$ and $\|x_m\|\nrightarrow 0$, by Theorem 0.1(a). We know that $\|x_m\|\to 0$ iff $I_\varphi(\lambda x_m)\to 0$ for every $\lambda>0$ [11]. So, there is $\lambda>1$ such that $I_\varphi(x_m)\to 0$ and $I_\varphi(\lambda x_m)\nrightarrow 0$. We can always find λ being arbitrarily close to one. Then, let $\lambda\in(1,1/r_0)$, where $r_0\in[1/2,1)$ is such that $M_0=\inf_n\varphi_n(r_0)>0$. The existence of such a number r_0 results from (*) and Lemma 1. Suppose, without loss of generality, that $I_\varphi(x_m)<M_0$ and $I_\varphi(\lambda x_m)\geqslant\varepsilon$ for each $m\in\mathbb{N}$ and some $\varepsilon\in(0,1)$. Now we find subsets N_m of $\mathbb{N}$ such that

$$\varepsilon/2\leqslant I_\varphi(\lambda x_m\chi_{N_m})\leqslant 1 \tag{10.2}$$

for each $m\in\mathbb{N}$. Indeed, since $I_\varphi(x_m)<M_0$, $\varphi_n(|u_{nm}|)<M_0$ for all $n\in\mathbb{N}$, where $x_m=(u_{nm})$. We have $|u_{nm}|<r_0$ for $n\in\mathbb{N}$, by the definition of M_0. Hence $\varphi_n(\lambda|u_{nm}|)<\varphi_n(\lambda r_0)\leqslant 1$. If there exists an index k such that $\varphi_k(\lambda|u_{km}|)\geqslant\varepsilon/2$ then we put $N_m=\{k\}$. If it is not true then $\varphi_k(\lambda|u_{km}|)+\varphi_l(\lambda|u_{lm}|)<2(\varepsilon/2)<1$ for each pair (k,l), $k\neq l$. We put $N_m=\{k,l\}$ if $\varphi_k(\lambda|u_{km}|)+\varphi_l(\lambda|u_{lm}|)\geqslant\varepsilon/2$ for any pair (k,l). Continuing this process we will find N_m satisfying (10.2), because $I_\varphi(\lambda x_m)\geqslant\varepsilon$. If we take

$$y_m=x_m\chi_{N_m},\qquad \bar{y}_m=\lambda x_m\chi_{N_m},$$

we have $I_\varphi(y_m)\leqslant 1$, $I_\varphi(\bar{y}_m)\leqslant 1$, by (10.2). Moreover, $I_\varphi(\bar{y}_m-y_m)=I_\varphi((\lambda-1)x_m\chi_{N_m})\leqslant(\lambda-1)I_\varphi(x_m)\to 0$, $m\to\infty$, because $\lambda-1\leqslant 1$. However, $|I_\varphi(\bar{y}_m)-I_\varphi(y_m)|=I_\varphi(\lambda x_m\chi_{N_m})-I_\varphi(x_m\chi_{N_m})\geqslant\varepsilon/4$ for large m, because $I_\varphi(x_m)\to 0$ and the condition (10.2) holds. This shows again that (10.1) cannot be fulfilled. In this way we have proved the necessity of the conditions (*), δ_2 and $\varphi_n(u)=0$ iff $u=0$ for satisfying (10.1).

Now, suppose the function φ satisfies (*), δ_2 and each φ_n vanishes only at zero. First, we will show the following:

> for each $d\in(0,1)$ and $\varepsilon>0$ there exists $\delta>0$ such that $I_\varphi(x)\leqslant 1$, $I_\varphi(y)\leqslant d$ and $I_\varphi(x-y)<\delta$ imply $|I_\varphi(x)-I_\varphi(y)|<\varepsilon$ for $x,y\in l_\varphi$. (10.3)

Indeed, by the assumed condition (*) and Lemma 9, $\|y\|\leqslant d_1$ for some $d_1\in(0,1)$. It is evident that $\|x\|\leqslant 1$. Let $\delta_1>0$ be the constant from Lemma 8 chosen for ε and d_1 in place of d. We find $\delta>0$ such that $I_\varphi(z)\leqslant\delta$ implies $\|z\|\leqslant\delta_1$ for $z\in l_\varphi$, by Theorem 0.1. So, if $I_\varphi(x)\leqslant 1$, $I_\varphi(y)\leqslant d$, and $I_\varphi(x-y)\leqslant\delta$ then $\|x\|\leqslant 1$, $\|y\|\leqslant d_1$ and $\|x-y\|<\delta_1$.

Hence $|I_\varphi(x)-I_\varphi(y)|<\varepsilon$ by Lemma 8, because I_φ satisfies the assumptions of f on $X=l_\varphi$.

Further, let $x=(u_n)$, $y=(v_n)$ and $I_\varphi(x)\leqslant 1$, $I_\varphi(y)\leqslant 1$. We will investigate a few cases.

First, let $\varphi_m(|u_m|)>1/2$ and $\varphi_m(|v_m|)>1/2$ for some index m. Let $\delta'>0$ from (10.3) be chosen for $d=\frac{1}{2}$ and $\varepsilon/2$. So, if $\sum_{n\neq m}\varphi_n(|u_n-v_n|)<\delta'$ then

$$\left|\sum_{n\neq m}\varphi_n(|u_n|)-\sum_{n\neq m}\varphi_n(|v_n|)\right|<\varepsilon/2, \tag{10.4}$$

because $\sum_{n\neq m}\varphi_n(|u_n|)<\frac{1}{2}$ and $\sum_{n\neq m}\varphi_n(v_n)<\frac{1}{2}$. Taking $\delta_0>0$ from Lemma 3 chosen for $\varepsilon/2$ we put $\delta''=\inf_n\varphi_n(\delta_0)$. We have $\delta''>0$, by our assumptions and Lemma 2. Moreover, if $\varphi_n(|u-v|)<\delta''$ then $|u-v|<\delta_0$ and hence

$$|\varphi_n(u)-\varphi_n(v)|<\varepsilon/2 \tag{10.5}$$

for all $n\in\mathbb{N}$. Let us put $\delta=\min(\delta',\delta'')$. If $I_\varphi(x-y)<\delta$ then $\sum_{n\neq m}\varphi_n(|u_n-v_n|)<\delta'$ and $\varphi_m(|u_m-v_m|)<\delta''$. Hence and by (10.4) and (10.5) we get $|I_\varphi(x)-I_\varphi(y)|<\varepsilon$.

Now, let $\varphi_m(|u_m|)>\frac{1}{2}$ and $\varphi_k(|v_k|)>\frac{1}{2}$ for some indices m, k, $m\neq k$. Let δ from (10.3) be chosen for $d=\frac{1}{2}$ and $\varepsilon/3$. Since $\sum_{n\neq m,k}\varphi_n(|v_n|)<\frac{1}{2}$, $\varphi_m(|v_m|)<\frac{1}{2}$, $\varphi_k(|u_k|)<\frac{1}{2}$, so

$$\begin{aligned}|I_\varphi(x)-I_\varphi(y)|\leqslant&\left|\sum_{n\neq m,k}\varphi_n(|u_n|)-\sum_{n\neq m,k}\varphi_n(|v_n|)\right|\\&+|\varphi_k(|v_k|)-\varphi_k(|u_k|)|\\&+|\varphi_m(|u_m|)-\varphi_m(|v_m|)|,\end{aligned}$$

if $I_\varphi(x-y)<\delta$, by (10.3).

Finally, let $\varphi_n(|u_n|)\leqslant\frac{1}{2}$ for all $n\in\mathbb{N}$. Since $I_\varphi(x)\leqslant 1$ so we find subsets N_1, N_2 of $\mathbb{N}$ such that $\mathbb{N}=N_1\cup N_2$, $N_1\cap N_2=\varnothing$ and $I_\varphi(x\chi_{N_1})\leqslant\frac{1}{2}$ and $I_\varphi(x\chi_{N_2})\leqslant\frac{3}{4}$. If we take δ from (10.3) for $d=\frac{3}{4}$ and $\varepsilon/2$ then we have

$$\begin{aligned}|I_\varphi(x)-I_\varphi(y)|\leqslant& I_\varphi(x\chi_{N_1})-I_\varphi(y\chi_{N_1})|\\&+|I_\varphi(x\chi_{N_2})-I_\varphi(y\chi_{N_2})|<\varepsilon\end{aligned}$$

for x, y satisfying $I_\varphi(x-y)<\delta$.

In all cases considered the number δ is dependent only on ε and the function φ. This remark ends the proof.

11. Proposition. *The space l_φ is uniformly rotund if and only if the following conditions are satisfied*:

(1) *the function* φ *satisfies the condition* δ_2,

(2) *the function* φ *satisfies the condition* (*),

(3) *each function* φ_n *vanishes only at zero,*

(4) *the modular* I_φ *is uniformly rotund.*

Proof. Let the space l_φ be uniformly rotund. Then l_φ is rotund and hence the function φ satisfies the conditions (1) and (3) (see Theorem 0.2).

Now, assume (2) is not satisfied. Then there exists a constant $\varepsilon \in (0, 1)$ and sequences (δ_n); $(m_n) \subset \mathbb{N}$, $(u_n) \subset (0, +\infty)$ such that $0 < \delta_n \downarrow 0$, $m_1 < m_2 < \cdots$, $\varphi_{m_n}(u_n) < \varepsilon$ and $\varphi_{m_n}((1+\delta_n)\,u_n) > 1$. Put $m_n = n$, without loss of generality. Let l_n be positive numbers such that $\varphi_n(l_n u_n) = 1$. Since $l_n \in (1, 1+\delta_n)$ and $\|u_n e_n\| = l_n^{-1}$, so

$$\|u_n e_n\| \to 1 \tag{11.1}$$

when $n \to \infty$, Let us take $\gamma_n \in (0, +\infty)$ in such a way that $\varphi_n(\gamma_n u_n) = (1+\varepsilon)/2$. Then $\gamma_n \to 1$, because $\gamma_n \in (1, 1+\delta_n)$. Let $v_n = \gamma_n u_n$, $w_n = 2u_n - v_n$. Moreover, let $s_n \in (0, \infty)$ be such that $\varphi_n(s_n) = (1-\varepsilon)/2$. Putting

$$x_n = v_{2n} e_{2n} + s_{2n+1} e_{2n+1}, \qquad y_n = w_{2n} e_{2n}$$

we have

$$I_\varphi(x_n) = \varphi_{2n}(v_{2n}) + \varphi_{2n+1}(s_{2n+1}) = (1+\varepsilon)/2 + (1-\varepsilon)/2 = 1,$$
$$I_\varphi(y_n) = \varphi_{2n}(2u_n - v_n) = \varphi_{2n}((2-\gamma_n)\,u_n) < 1,$$

for all $n \in \mathbb{N}$, because $2 - \gamma_n < 1$. Moreover, $I_\varphi(x_n - y_n) = \varphi_{2n}(|v_{2n} - w_{2n}|) + \varphi_{2n+1}(s_{2n+1}) \geqslant (1-\varepsilon)/2$ for all $n \in \mathbb{N}$. But $(x_n + y_n)/2 = u_{2n} e_{2n} + (s_{2n+1}/2)\, e_{2n+1} \geqslant u_{2n} e_{2n}$, which implies $\|(x_n + y_n)/2\| \geqslant \|u_{2n} e_{2n}\| \to 1$, by the monotonicity of the norm and (11.1). This contradicts the uniform rotundity of l_φ.

Now, let $I_\varphi(x) = 1$, $I_\varphi(y) = 1$, and $I_\varphi(x-y) \geqslant \varepsilon$. Hence and by the well-known properties of the Luxemburg norm we have $\|x\| = 1$, $\|y\| = 1$, and $\|x - y\| \geqslant \varepsilon_1(\varepsilon)$ for some $\varepsilon_1(\varepsilon) > 0$. Then $\|(x+y)/2\| \leqslant 1 - p(\varepsilon)$ for some $p(\varepsilon) \in (0, 1)$. However, $I_\varphi((x+y)/2) \leqslant \|(x+y)/2\|$, which shows the uniform rotundity of the modular I_φ, i.e., the condition (4).

Supposing the conditions (1)–(4), let us take $x, y \in l_\varphi$ such that $\|x\| = 1$, $\|y\| = 1$, and $\|x - y\| \geqslant \varepsilon$. There exists $\varepsilon_1(\varepsilon) > 0$ such that $I_\varphi(x-y) \geqslant \varepsilon_1(\varepsilon)$, by (1), (3), and Theorem 0.1. We also have $I_\varphi(x) = 1$ and $I_\varphi(y) = 1$, by Theorem 0.1(b). So, there exists $p_1(\varepsilon) \in (0, 1)$ such that $I_\varphi((x+y)/2) \leqslant 1 - p_1(\varepsilon)$, by the assumption (4). Now, by virtue of (2) and Lemma 9 we find $p(\varepsilon) \in (0, 1)$ satisfying $\|(x+y)/2\| \leqslant 1 - p(\varepsilon)$, which ends the proof of this theorem.

1.2. *Remark.* Equivalently, under assumptions (1)–(3) of the above proposition we can put $I_\varphi(x)\leqslant 1$, $I_\varphi(y)\leqslant 1$ instead of $I_\varphi(x)=1$, $I_\varphi(y)=1$ in the definition of uniform rotundity of the modular I_φ. It can be shown by the same technique as that in the above proof.

12. LEMMA. *If φ satisfies the condition δ_2 and all φ_n vanish only at zero then the modular I_φ is uniformly rotund iff*

for each $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that if $I_\varphi(x)=I_\varphi(y)=1$, where $x=(u_n)$, $y=(v_n)$ are arbitrary with $u_n\geqslant 0$, $v_n\geqslant 0$, and $I_\varphi(x-y)\geqslant\varepsilon$, then $I_\varphi((x+y)/2)\leqslant 1-\delta(\varepsilon)$. (12.1)

Proof. Let us suppose the condition (12.1) is fulfilled and take x,y such that $I_\varphi(x)=I_\varphi(y)=1$ and $I_\varphi(x-y)\geqslant\varepsilon$. There exists an index m such that $I_\varphi((x-y)\,\chi_{\mathbb{N}\setminus\{m\}})\geqslant\varepsilon/2$. Let

$$N_0=\{n\in\mathbb{N}\setminus\{m\}\colon u_nv_n<0\}$$
$$N_1=\{n\in N_0\colon |u_n|\leqslant|v_n|\}$$
$$N_2=\{n\in N_0\colon |u_n|>|v_n|\}$$

and put

$$\bar{u}_n=\begin{cases}\tilde{u}_n, & n=m\\ 0, & n\in N_1\\ |u_n|, & \text{otherwise},\end{cases}\qquad \bar{v}_n=\begin{cases}\tilde{v}_n, & n=m\\ 0, & n\in N_2\\ |v_n|, & \text{otherwise},\end{cases}$$

where $\tilde{u}_m,\tilde{v}_m$ are nonnegative numbers chosen in such a way that $\sum_{N_1}\varphi_n(|u_n|)+\varphi_m(|u_m|)=\varphi_m(\tilde{u}_m)$, $\sum_{N_2}\varphi_n(|v_n|)+\varphi_m(|v_m|)=\varphi_m(\tilde{v}_m)$. Denoting $\bar{x}=(\bar{u}_n)$, $\bar{y}=(\bar{v}_n)$ we have $I_\varphi(\bar{x})=I_\varphi(\bar{y})=1$ and $I_\varphi((x+y)/2)\leqslant I_\varphi((\bar{x}+\bar{y})/2)$, immediately. Moreover,

$$I_\varphi(\bar{x}-\bar{y})\geqslant\max(I_\varphi((x-y)/2\,\chi_{N_0}),\,I_\varphi((x-y)\,\chi_{\mathbb{N}\setminus\{m\}\setminus N_0}). \tag{12.2}$$

Choosing a constant k and a sequence (c_n) in the condition δ_2 so that they satisfy (0.3) we get

$$I_\varphi((x-y)/2\,\chi_{N_0})\geqslant 1/k\,I_\varphi((x-y)\,\chi_{N_0})-\varepsilon/4k. \tag{12.3}$$

Since $I_\varphi((x-y)\,\chi_{\mathbb{N}\setminus\{m\}})\geqslant\varepsilon/2$, so $I_\varphi(\bar{x}-\bar{y})\geqslant\varepsilon/4$ or $I_\varphi(\bar{x}-\bar{y})\geqslant\varepsilon/4k$, by (12.2) and (12.3).

The above lemma is very useful in the proof of the next theorem, because the investigation concerning uniform rotundity of I_φ can be limited to elements with nonnegative coefficients.

1. THEOREM. *The space l_φ is uniformly rotund if and only if the following conditions are satisfied:*

(1) *the function φ fulfills the condition δ_2;*

(2) *the function φ fulfills the condition (*);*

(3) *each functions φ_n vanishes only at zero;*

(4) *there exists a sequence $(a_n) \subset [0, 1]$ such that $\varphi_n(a_n) + \varphi_m(a_m) \geqslant 1$ for all $n \neq m$ and each φ_n is strictly convex on the interval $[0, a_n]$, respectively;*

(5) *the function φ is uniformly convex in the d-neighbourhood of zero, where $d = (1 - \inf_n \varphi_n(a_n)) \vee \frac{1}{2}$.*

Proof. Sufficiency. Let $I_\varphi(x) = I_\varphi(y) = 1$, $I_\varphi(x-y) \geqslant \varepsilon$ for some $\varepsilon \in (0, 1)$, where $x = (u_n)$, $y = (v_n)$ and u_n, v_n are nonnegative. Infimum expressing the number d is not necessarily attained. Denoting $\alpha = \inf_n \varphi_n(a_n)$ let φ_0 be any Young function strictly convex on $[0, \varphi_0^{-1}(\alpha)]$ and linear on $[\varphi_0^{-1}(\alpha), 1]$. We can isometrically imbed the space l_φ into $l_{\bar{\varphi}}$, where $\bar{\varphi} = (\varphi_n)_{n=0}^{\infty}$. So, in the sequel we assume that the infimum is attained and $\inf_n \varphi_n(a_n) = \varphi_1(a_1)$. Note that the function φ_1 is strictly convex on $[0, \varphi_1^{-1}(1-d)]$. In particular, if $d = 1$ then φ_1 is linear on some neighbourhood of zero. There exist at most two indices m, k such that $\varphi_m(u_m) > d$ and $\varphi_k(v_k) > d$. First we shall estimate the expression $I_\varphi((x+y)/2)$ in the following two situations.

(A) There exist $j \in \mathbb{N}$, $p \in (0, 1)$, $b \in (0, 1)$ such that $h_j(u_j, v_j) \leqslant 1 - p$ and $\varphi_j(u_j \vee v_j) \geqslant b$.

(B) There exist constants $\delta > 0$, $c \in (0, d]$ and a subset N_0 of $\mathbb{N}$ such that $\varphi_n(u_n \vee v_n) \leqslant c$ for $n \in N_0$, $\sum_{N_0} \varphi_n(|u_n - v_n|) \geqslant \delta$ and φ_n are strictly convex on the inverse image of $[0, c]$, respectively, for $n \in N_0$.

ad. (A) If $h_j(u_j, v_j) \leqslant 1 - p$ and $\varphi_j(u_j \vee v_j) \geqslant b$ then

$$I_\varphi((x+y)/2) \leqslant 1 - (p/2)(\varphi_j(u_j) + \varphi_j(v_j)) \leqslant 1 - pb/2. \tag{1.1}$$

ad. (B) Let

$$E = \{n \in N_0 \colon |u_n - v_n| \geqslant (\delta/8)(u_n \vee v_n) \text{ and } u_n \vee v_n \in (c_n, \varphi_n^{-1}(c)]\},$$

where (c_n) is the sequence from Lemma 6 chosen for $\delta/8$, c instead of ε, d. So, by virtue of Remark 1.1 there exists $p = p(\delta, c) \in (0, 1)$ such that $h_n(u_n, v_n) \leqslant 1 - p$ for $n \in E$. Hence immediately

$$I_\varphi((x+y)/2) \leqslant 1 - (p/2)(I_\varphi(x\chi_E) + I_\varphi(y\chi_E)), \tag{1.2}$$

because $I_\varphi(x) = I_\varphi(y) = 1$. However, $\varphi_n(|u_n - v_n|) \leqslant (\delta/8)(\varphi_n(u_n) + \varphi_n(v_n))$

or $\varphi_n(|u_n - v_n|) \leqslant \varphi_n(2(u_n \vee v_n)) \leqslant \varphi_n(2c_n)$ for $n \in N_0 \backslash E$. So, $\sum_{N_0 \backslash E} \varphi_n(|u_n - v_n|) \leqslant (\delta/8)(I_\varphi(x) + I_\varphi(y)) + \sum_{n=1}^{\infty} \varphi_n(2c_n) \leqslant \delta/4 + \delta/8 < \delta/2$. Then $\sum_E \varphi_n(|u_n - v_n|) \geqslant \delta/2$, by the assumption $\sum_{N_0} \varphi_n(|u_n - v_n|) \geqslant \delta$. Applying the condition δ_2 with the constant k and the sequence (c_n) from Lemma 6 we get

$$\begin{aligned}\delta/2 < \sum_E \varphi_n(|u_n - v_n|) &\leqslant (k/2)\Big(\sum_E \varphi_n(u_n) + \sum_E \varphi_n(v_n)\Big) + \sum_E \varphi_n(2c_n)\\ &\leqslant (k/2)(I_\varphi(x\chi_E) + I_\varphi(y\chi_E)) + \delta/8.\end{aligned}$$

Hence $I_\varphi(x\chi_E) + I_\varphi(y\chi_E) \geqslant 3\delta/4k$, which in connection with (1.2) gives the estimation

$$I_\varphi((x+y)/2) \leqslant 1 - 3p\delta/8k. \tag{1.3}$$

Further we shall show that the estimation of $I_\varphi((x+y)/2)$ is always of the type (A) or (B). We shall consider two main cases.

(I) Let $\varphi_n(u_n) \leqslant d$ and $\varphi_n(v_n) \leqslant d$ for all $n \in \mathbb{N}$, $n \neq 1$. If $\varphi_1(|u_1 - v_1|) < \varepsilon/2$ then we have (B) with $\delta = \varepsilon/2$, $c = d$ and $N_0 = \mathbb{N} \backslash \{1\}$. If $\varphi_1(|u_1 - v_1|) \geqslant \varepsilon/2$, then by convexity of φ_1 we have $|\varphi_1(u_1) - \varphi_1(v_1)| \geqslant \varepsilon/2$ and $|\sum_{n \neq 1} \varphi_n(u_n) - \sum_{n \neq 1} \varphi_n(v_n)| = |1 - \varphi_1(u_1) - 1 + \varphi_1(v_1)| \geqslant \varepsilon/2$. Applying Proposition 10 with $\varepsilon/2$ instead of ε we will find $\delta > 0$ dependent only on ε such that $\sum_{n \neq 1} \varphi_n(|u_n - v_n|) \geqslant \delta$. So, we also get the case (B) with $c = d$ and $N_0 = \mathbb{N} \backslash \{1\}$.

If the case (I) does not hold then we can write, without loss of generality, the following.

(II) There exists an index $k \neq 1$ such that $\varphi_k(v_k) > d$. It is evident that we can put $d < 1$. In the sequel let i be a natural number such that $i > 2$ and $\varepsilon/2^i \leqslant 1 - d$.

—Let $\varphi_k(u_k) \leqslant d - \varepsilon/2^i$. We shall find $p \in (0, 1)$ such that $h_k(u_k, v_k) \leqslant 1 - p$, by Lemma 7 applied to $d - \varepsilon/2^i$, d in place of α, β. So, it is the case (A) with $j = k$, $b = d$.

—Let $\varphi_k(u_k) > d - \varepsilon/2^i$ and $\varphi_n(u_n) < 1 - d$ for each $n \neq k$. We shall show that $\sum_{n \neq k} \varphi_n(|u_n - v_n|) > \delta$ for some $\delta = \delta(\varepsilon)$. Indeed, if $\varphi_k(|u_k - v_k|) \geqslant \varepsilon/2$ then $|\varphi_k(u_k) - \varphi_k(v_k)| \geqslant \varphi_k(|u_k - v_k|) \geqslant \varepsilon/2$. Hence $|\sum_{n \neq k} \varphi_n(u_n) - \sum_{n \neq k} \varphi_n(v_n)| = |\varphi_k(u_k) - \varphi_k(v_k)| \geqslant \varepsilon/2$. Therefore, we find a suitable δ by Proposition 10. So, we get the situation (B) for $c = 1 - d$, $N_0 = \mathbb{N} \backslash \{k\}$.

—Let $\varphi_k(u_k) > d - \varepsilon/2^i$ and $\varphi_m(u_m) \geqslant 1 - d$ for some $m \neq k$. Then

$$d - \varepsilon/2^i < \varphi_k(u_k) \leqslant d \quad \text{and} \quad 1 - d \leqslant \varphi_m(u_m) < (1-d) + \varepsilon/2^i. \tag{1.4}$$

Moreover, we have $\varphi_k(v_k) > d$ and $\varphi_m(v_m) < 1-d$, by the assumption (II). If

$$\varphi_k(v_k) > d + \varepsilon/2^i \qquad \text{or} \qquad \varphi_m(v_m) < (1-d) - \varepsilon/2^i, \tag{1.5}$$

then applying Lemma 7 with d, $d-\varepsilon/2^i$ or $(1-d)-\varepsilon/2^i$, $1-d$ in place of α, β, we find $p \in (0, 1)$ such that $h_j(u_j, v_j) \leqslant 1-p$ for $j=k$ or $j=m$. This is the case (A) with $b=d$ or $b=1-d$. Contrary to (1.5), we have

$$d < \varphi_k(v_k) \leqslant d + \varepsilon/2^i \qquad \text{and} \qquad (1-d) - \varepsilon/2^i \leqslant \varphi_m(v_m) < 1-d. \tag{1.6}$$

Then $\varphi_j(|u_j - v_j|) \leqslant |\varphi_j(u_j) - \varphi_j(v_j)| \leqslant d + \varepsilon/2^i - d + \varepsilon/2^i = \varepsilon/2^{i-1}$ for $j=k, m$ by (1.4) and (1.6). Hence $\sum_{n \neq m,k} \varphi_n(|u_n - v_n|) \geqslant \varepsilon - \varphi_m(|u_m - v_m|) - \varphi_k(|u_k - v_k|) \geqslant (1 - 1/2^{i-2})\,\varepsilon$. Putting $\delta = (1-1/2^{i-2})\,\varepsilon$, $c = 1-d$, $N_0 = \mathbb{N} \setminus \{m, k\}$ we have the situation (B).

In all the cases considered we obtained the estimation of $I_\varphi((x+y)/2)$ expressed by inequalities (1.1) and (1.3), where constants p, b, δ, k are dependent only on ε and the function φ. So we showed uniform rotundity of l_φ, by Proposition 11 and Lemma 12.

Necessity. The conditions (1)–(4) are satisfied, by Proposition 11 and Theorem 0.2. To prove (5), let us note that uniform convexity of φ in the d-neighbourhood of zero is equivalent to the following condition.

> For every $a \in (0, 1)$ there exists $\delta \in (0, 1)$ such that $\sum_{n=1}^{\infty} \varphi_n(u_n(\delta, a)) < \infty$, were $u_n(\delta, a) = \sup\{u \in [0, \varphi_n^{-1}(d)]: h_n(u, au) \geqslant 1 - \delta\}$. (1.7)

Assuming that (5) is not satisfied we get

$$\sum_{n=1}^{\infty} \varphi_n(u_{nk}) = \infty \tag{1.8}$$

for each $k \in \mathbb{N}$, where $u_{nk} = u_n(\delta_k, a) \in [0, \varphi_n^{-1}(d)]$, a is some constant from the interval (0, 1) and (δ_k) is a sequence included in (0,1) such that $\delta_k \downarrow 0$. By definition of the sequence $(u_n(\delta, a))$, the inequality

$$h_n(u_{nk}, au_{nk}) \geqslant 1 - \delta_k \tag{1.9}$$

holds, for each $n,k \in \mathbb{N}$. In the sequel we shall consider two cases.

(I) There exists $b \in (0, d)$ such that $\overline{\lim}_{k \to \infty} \sup_{n \geqslant m} \varphi_n(u_{nk}) > b$ for each $m \in \mathbb{N}$. Hence, one can find increasing subsequences (n_j), (k_j) of $\mathbb{N}$ such that $\varphi_{n_j}(u_{n_j k_j}) \in (b, d]$ for each $j \in \mathbb{N}$. For simplicity, we put j and v_j in place of n_j and $u_{n_j k_j}$. So $\varphi_j(v_j) \in (b, d]$ for each $j \in \mathbb{N}$.

Assume for the moment that $\varphi_j(v_j) \leqslant \frac{1}{2}$ except for at most a finite number of indices. Without loss of generality, we put $\varphi_j(v_j) \leqslant \frac{1}{2}$ for every $j \in \mathbb{N}$. We can also choose a monotone infinite subsequence of $(\varphi_j(v_j)-\varphi_j(av_j))$. Therefore we can suppose, e.g., this whole sequence to be nondecreasing, i.e., $\varphi_j(v_j)+\varphi_{j+1}(av_{j+1})\leqslant\varphi_{j+1}(v_{j+1})+\varphi_j(av_j)$. Then there exists $a_j\in[a, 1)$ such that

$$\varphi_{2j}(v_{2j})+\varphi_{2j+1}(a_jv_{2j+1})=\varphi_{2j+1}(v_{2j+1})+\varphi_{2j}(av_{2j}), \tag{1.10}$$

for every $j\in\mathbb{N}$. The expressions from both sides of the above equality are less than one, because $\varphi_j(v_j)+\varphi_i(av_i)<1$ for each $i, j\in\mathbb{N}$. Therefore

$$\varphi_{2j}(v_{2j})+\varphi_{2j+1}(a_jv_{2j+1})+\varphi_1(c_j)=1 \tag{1.11}$$

for some $c_j>0$. Let

$$x_j=v_{2j}e_{2j}+a_jv_{2j+1}e_{2j+1}+c_je_1$$
$$y_j=av_{2j}e_{2j}+v_{2j+1}e_{2j+1}+c_je_1.$$

We have $I_\varphi(x_j)=I_\varphi(y_j)=1$, by (1.10) and (1.11). Moreover, $I_\varphi((x_j-y_j)/(1-a))\geqslant\varphi_{2j}(v_{2j})>b$ for each $j\in\mathbb{N}$. Then $I_\varphi(x_j-y_j)\geqslant c$ for some $c>0$ and each $j\in\mathbb{N}$, by the condition δ_2. By virtue of (1.9) and definition of (v_j), $h_j(v_j, av_j)\geqslant 1-\delta_{k_j}$. Therefore and by the second property of h_j considered in Lemma 0.3 $h_{2j+1}(v_{2j+1}, a_jv_{2j+1})\geqslant h_{2j+1}(v_{2j+1}, av_{2j+1})\geqslant 1-\delta_{k_{2j+1}}$ is satisfied. Hence

$$\begin{aligned} I_\varphi((x_j+y_j)/2)\geqslant{}&(1-\delta_{k_{2j}})(\varphi_{2j}(v_{2j})+\varphi_{2j}(av_{2j}))/2\\ &+(1-\delta_{k_{2j+1}})(\varphi_{2j+1}(v_{2j+1})+\varphi_{2j+1}(a_jv_{2j+1}))/2\\ &+\varphi_1(c_j)\geqslant 1-\delta_{k_{2j}}\to 1, \end{aligned}$$

when $j\to\infty$, by monotone convergence of (δ_{k_j}) to zero and (1.10) and (1.11).

Now let $\varphi_j(v_j)>\frac{1}{2}$ for an infinite number of indices. For simplicity we put $\varphi_j(v_j)>\frac{1}{2}$ for every $j\in\mathbb{N}$. However, $\varphi_j(v_j)\leqslant d$, so $d>\frac{1}{2}$. Then, by (5), $\inf_n\varphi_n(a_n)<\frac{1}{2}$. It implies, by virtue of (4), that the infimum must be attained. So, we can put $\inf_n\varphi_n(a_n)=\varphi_1(a_1)$ and $d=1-\varphi_1(a_1)$. The function φ_1 is linear on some interval $[a_1, \bar{a}_1]$. Let $b_1\in(a_1, \bar{a}_1)$ be such that $\varphi_1(b_1)-\varphi_1(a_1)\leqslant(1-a)\,b$. Hence

$$\begin{aligned} &\varphi_1((a_1+b_1)/2)=(\varphi_1(a_1)+\varphi_1(b_1)/2\\ &\varphi_j(av_j)+\varphi_1(b_1)\leqslant\varphi_j(v_j)+\varphi_1(a_1) \end{aligned} \tag{1.12}$$

holds for each $j\in\mathbb{N}$ immediately, because $(1-a)\,b\leqslant(1-a)\,\varphi_j(v_j)\leqslant\varphi_j(v_j)-\varphi_j(av_j)$, by convexity of φ_j. So, there exist $a_j\in[a, 1)$ such that

$$\varphi_j(a_jv_j)+\varphi_1(b_1)=\varphi_j(v_j)+\varphi_1(a_1). \tag{1.13}$$

Moreover, we find $c_j \geqslant 0$ such that

$$\varphi_j(v_j) + \varphi_1(a_1) + \varphi_2(c_j) = 1, \tag{1.14}$$

because $\varphi_j(v_j) + \varphi_1(a_1) \leqslant d + \varphi_1(a_1) \leqslant 1$. Let

$$x_j = a_1 e_1 + c_j e_2 + v_j e_j$$
$$y_j = b_1 e_1 + c_j e_2 + a_j v_j e_j.$$

We have $I_\varphi(x_j) = I_\varphi(y_j) = 1$, $I_\varphi(x_j - y_j) \geqslant \varphi_1(b_1 - a_1) > 0$ for each $j \geqslant 3$. However, $h_j(v_j, a_j v_j) \geqslant h_j(v_j, av_j) \geqslant 1 - \delta_{k_j}$, by the property of h_j and (1.9), and so

$$\begin{aligned} I_\varphi((x_j + y_j)/2) &\geqslant \varphi_1(a_1)/2 + \varphi_1(b_1)/2 + \varphi_2(c_j) \\ &\quad + (1 - \delta_{k_j})(\varphi_j(v_j) + \varphi_j(a_j v_j))/2 \\ &\geqslant 1 - (\delta_{k_j}/2)(\varphi_j(v_j) + \varphi_j(a_j v_j)) \\ &\geqslant 1 - \delta_{k_j} \to 1, \end{aligned}$$

when $j \to \infty$, by (1.12), (1.13), and (1.14).

(II) Contrary to (I), for every $b \in (0, d)$ there exists $m \in \mathbb{N}$ such that $\overline{\lim}_{k\to\infty} \sup_{n \geqslant m} \varphi_n(u_{nk}) \leqslant b$. Hence we find subsequences (m_j), (k_j) of $\mathbb{N}$ such that (k_j) is increasing and

$$\varphi_n(u_{nk_j}) \leqslant 1/2^{j+1} \tag{1.15}$$

for each $j \in \mathbb{N}$, $n \geqslant m_j$. Moreover, it is known that

$$\sum_{n=1}^{\infty} \varphi_n(u_{nk_j}) = \infty, \qquad \sum_{n=1}^{\infty} \varphi_n(au_{nk_j}) = \infty, \tag{1.16}$$

by (1.8) and the condition δ_2. We shall show that for each $j \in \mathbb{N}$ there exist two disjoint subsets N_{1j}, N_{2j} of $\mathbb{N}$ such that

$$1 - 1/2^{j-1} \leqslant \sum_{N_{1j}} \varphi_n(u_{nk_j}) + \sum_{N_{2j}} \varphi_n(au_{nk_j}) \leqslant 1 - 1/2^j, \tag{1.17}$$

$$\left| \sum_{N_{1j}} (\varphi_n(u_{nk_j}) - \varphi_n(au_{nk_j})) - \sum_{N_{2j}} (\varphi_n(u_{nk_j}) - \varphi_n(au_{nk_j})) \right| < 1/2^{j+1}, \tag{1.18}$$

putting $\sum_\varnothing = 0$. Indeed, let j be fixed at present. We put $m_j \in N_{1j}$. We have

$$\varphi_n(u_{nk_j}) - \varphi_n(au_{nk_j}) \leqslant 1/2^{j+1},$$

for all $n \geq m_j$, by (1.15). If the sum of the left side of the above inequality for $n = m_j$ and $n = m_j + 1$ is less than or equal to $1/2^{j+1}$ then we put $m_j + 1 \in N_{1j}$. If this sum is greater than $1/2^{j+1}$ then we put $m_j + 1 \in N_{2j}$. We have always $|(\varphi_{m_j}(u_{m_j k_j}) - \varphi_{m_j}(au_{m_j k_j})) - (\varphi_{m_j+1}(u_{m_j+1\, k_j}) - \varphi_{m_j+1}(au_{m_j+1\, k_j}))| \leq 1/2^{j+1}$. If additionally,

$$1 - 1/2^{j-1} \leq \varphi_{m_j}(u_{m_j k_j}) + \varphi_{m_j+1}(au_{m_j+1\, k_j}) \leq 1 - 1/2^j,$$

then we put $N_{1j} = \{m_j\}$ and $N_{2j} = \{m_j + 1\}$. In the opposite case we continue this process finding sets N_{1j}, N_{2j} in a finite number of steps, because (1.15) and (1.16) hold. Let

$$x_j = \sum_{N_{1j}} u_{nk_j} e_n + \sum_{N_{2j}} au_{nk_j} e_n$$

$$y_j = \sum_{N_{1j}} au_{nk_j} e_n + \sum_{N_{2j}} u_{nk_j} e_n.$$

We have $I_\varphi(x_j) \leq 1$ and $I_\varphi(y_j) \leq 1$, by (1.17) and (.18). Moreover, $I_\varphi((x_j - y_j)/(1-a)) = \sum_{N_{1j} \cup N_{2j}} \varphi_n(u_{nk_j}) \geq \frac{1}{2}$, by (1.17), and so $I_\varphi(x_j - y_j) \geq c$ for some $c > 0$ and all $j \in \mathbb{N}$. However,

$$\begin{aligned} I_\varphi((x_j + y_j)/2) &= \sum_{N_{1j} \cup N_{2j}} \varphi_n((u_{nk_j} + au_{nk_j})/2) \\ &\geq (1 - \delta_{k_j}) \sum_{N_{1j} \cup N_{2j}} (\varphi_n(u_{nk_j}) + \varphi_n(au_{nk_j}))/2. \end{aligned}$$

The right side of the inequality tends to 1, by (1.9), (1.17), and (1.18). We have shown that if φ does not satisfy the condition (5) then the modular I_φ is not uniformly rotund, in all the cases considered above. Then, by Proposition 11 and Remark 1.2, the necessity of the condition (5) is shown, which ends the proof.

In particular, if all φ_n are equal, the known criterion of uniform rotundity of Orlicz sequence spaces (Theorem 7 in [7]) is easily obtained from the above theorem.

Let (p_n) be a sequence of real numbers $p_n \in [1, \infty)$. By $l(\{p_n\})$ we denote the Nakano space [12]. Then the space $l(\{p_n\})$ is the set of all real sequences $x = (u_n)$ such that $\sum_{n=1}^\infty (1/p_n)|\lambda u_n|^{p_n} < \infty$ for some $\lambda > 0$ dependent on x. Indeed, $l(\{p_n\})$ is the Musielak–Orlicz sequence space l_φ, endowed with Luxemburg norm, if we put $\varphi_n(u) = (1/p_n)\, u^{p_n}$, $u \in \mathbb{R}_+$. This space we can isometrically transform in such a way that $\varphi_n(u) = u^{p_n}$ if $u \in [0, 1]$, $\varphi_n(u) = u$ if $u > 1$, as we have shown at the beginning of this paper. Sundaresan in [12] has given a sufficient condition and a slightly weaker necessary condition for uniform rotundity of $l(\{p_n\})$. We shall show that a criterion of uniform rotundity of $l(\{p_n\})$ results from our main theorem.

2. THEOREM. *The space $l(\{p_n\})$ is uniformly rotund if and only if*

$$1 < \varliminf_{n\to\infty} p_n \leqslant \varlimsup_{n\to\infty} p_n < \infty \text{ and } p_n = 1 \text{ for at most one index } n. \quad (2.1)$$

Proof. Let us note $\varphi_n(u) = u^{p_n}$, $u \in [0, 1]$. Suppose the condition (2.1) is satisfied. There exist p and q such that $1 < p \leqslant q < \infty$ and $p_n \leqslant q$ for all $n \in \mathbb{N}$ and $p_n \geqslant p$ for almost all $n \in \mathbb{N}$. If $\varphi_n(u) = u^{p_n} \leqslant 1 - \varepsilon$ then $u \leqslant (1-\varepsilon)^{1/q}$ for $\varepsilon \in (0, 1)$. It is evident that $(1+\delta)^q (1-\varepsilon)^{1/q} < 1$ for some $\delta > 0$. Hence $\varphi_n((1+\delta) u) = (1+\delta)^{p_n} u^{p_n} \leqslant (1+\delta)^q u \leqslant (1+\delta)^q (1-\varepsilon)^{1/q} < 1$, if $\varphi_n(u) \leqslant 1-\varepsilon$. This shows that $\varphi = (\varphi_n)$ satisfies the condition (*). The condition δ_2 is also satisfied, because $\varphi_n(2u) \leqslant 2^{q+1} \varphi_n(u)$ for all $u \in \mathbb{R}_+$. It is enough to prove (5), because (3) and (4) are evident even though one of p_n is equal to 1. We have the inequalities (see (i_1) and (i_2) in [7])

$$((1+a)/2)^{p_n} \leqslant (1+a^{p_n})/2 - ((1-a)/2)^{p_n} \qquad \text{if } p_n \geqslant 2,$$

$$((1+a)/2)^{p_n} \leqslant (1+a^{p_n})/2 - (p_n(p_n-1)/2) \times ((1-a)/(1+a))^{2-p_n}((1-a)/2)^{p_n}, \qquad \text{if } p_n < 2,$$

for any number $a \in [0, 1)$. Hence it follows simply that

$$h_n(u, au) \leqslant 1 - ((1-a)/2)^q (1+a^q)/2 \qquad \text{for } p_n \geqslant 2,$$

$$h_n(u, au) \leqslant 1 - (p(p-1)/2)((1-a)/(1+a))^{2-p} \times ((1-a)/2)^2 2/(1+a^p) \qquad \text{for } p \leqslant p_n < 2,\ u \in [0, 1].$$

Therefore the condition (5) is fulfilled with $d = 1$ and $(c_n) = (1, 0, \ldots)$, putting $p_1 = 1$.

Let the space $l(\{p_n\})$ be uniformly rotund. Then the conditions of the previous theorem must be fulfilled. The existence of n, at most one, for which $p_n = 1$, follows easily from (4). Suppose $\varlimsup_{n\to\infty} p_n = \infty$. For simlicity we write $\lim_{n\to\infty} p_n = \infty$. If $u_n = \sqrt[p_n]{1-\varepsilon}$ then $u_n \to 1$ when $n \to \infty$. This contradicts the condition (*). Now, suppose $\varliminf_{n\to\infty} p_n = 1$. There is an infinite decreasing sequence (p_{n_i}) such that $p_{n_i} > 1$ and $\lim_{i\to\infty} p_{n_i} = 1$. Then

$$h_{n_i}(u, au) = ((1+a)/2)^{p_{n_i}} (2/(1+a^{p_{n_i}})) \to 1,$$

if $i \to \infty$, for all $u \in [0, 1]$ and $a \in [0, 1)$. Therefore the condition (5) cannot be fulfilled. This completes the proof.

Finally let us note that the case of atomless measure was considered in [8] for Orlicz spaces and in [5] for Musielak–Orlicz spaces.

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